D-Branes in the Euclidean AdS_3 and T-Duality

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Abstract We show that D-branes in the Euclidean AdS_3 can be naturally associated to the maximally isotropic subgroups of the Lu–Weinstein double of $SU(2)$. This picture makes very transparent the residual loop group symmetry of the D-brane configurations and gives also immediately the D-branes shapes and the σ-model boundary conditions in the de Sitter T-dual of the $SL(2, C)/SU(2)$ WZW model.

1 Introduction

An even-dimensional Lie group D equipped with a maximally Lorentzian biinvariant metric is called a Drinfeld double[1] [1, 2, 6] if it has at least two maximally isotropic Lie subgroups G_1 and G_2, not related by an inner automorphism in D. It was shown in [6], that the Drinfeld double D and an unipotent linear operator $\mathcal{E}$ on its Lie algebra $\mathcal{D}$ naturally define mutually dual closed string σ-models on the targets D/G_1 and D/G_2, respectively. By considering moreover an element $d \in D$ and another maximally isotropic subgroup M of D, the quadruple $(D, \mathcal{E}, d, M)$ defines a mutually dual pair of open string σ-models [7, 8]. In particular, the D-brane submanifolds of the targets D/G_1 and D/G_2 are, respectively, the coset projections π_{G_1} and π_{G_2} of $Md \subset D$ to D/G_1 and D/G_2.

We have recently shown [9], that for $D = SL(2, \mathbf{C})$, $G_1 = SU(2)$, $G_2 = SL(2, \mathbf{R})$ and an appropriate $\mathcal{E}$, the corresponding bulk σ-model on D/G_1 is nothing but the $SL(2, \mathbf{C})/SU(2)$ WZW model [10] describing strings in the Euclidean AdS_3, while the dual model living on D/G_2 captures the string dynamics in the three-dimensional de Sitter space. In this paper, we shall enlarge our discussion to incorporate the open string σ-models. We shall show, in particular, that all AdS_3 D-brane boundary conditions, recently considered by Ponsot,

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[1]Here we commit a little abuse of terminology. In fact, for the standardly defined Drinfeld double [1, 2], it must be moreover true that $Lie(D)$ is the (vector space) direct sum $Lie(G_1) + Lie(G_2)$. The latter condition was even present in the original version of the Poisson–Lie T-duality [3–5]. However, as it was shown later in [6], it can be released and the duality continues to take place.

Schomerus and Teschner [11], can be obtained by choosing appropriately the data d and M of the general construction indicated above. This means that we will be able to write down immediately also de Sitter duals of the AdS_3 D-branes.

From the technical point of view, this paper just makes explicit the general theory of D-branes T-duality [7, 8] for the particular case of the $SL(2, \mathbf{C})/SU(2)$ WZW model. We believe that it is worth working out this example for two reasons: (1) the $SL(2, \mathbf{C})/SU(2)$ WZW model plays an important role in the AdS/CFT correspondence therefore its duality structure should be of interest; (2) the model is conformal and has a loop group symmetry hence it is so far the best candidate for testing the quantum status of the Poisson–Lie T-duality.

The Poisson–Lie T-duality is relevant for the $SL(2, \mathbf{C})/SU(2)$ WZW model because the latter is Poisson–Lie symmetric [9]. It is less obvious, however, why the D-branes boundary conditions considered by Ponsot, Schomerus and Teschner [11] fits in the framework of the Poisson–Lie T-duality. In fact, the authors of [11] were guided by the requirement of the residual loop group symmetry of the open string σ-model. This condition gave them the shapes of the AdS_3 D-branes as well as B-fields encoding the boundary condition. On the other hand, the authors of [7, 8] were using the criterion of T-dualizability for finding the shape of D-branes and boundary conditions on them. Remarkably, those very differently looking criterions give the same D-branes for the $SL(2, \mathbf{C})/SU(2)$ WZW model. We shall offer the explanation of this fact based on the association of the D-branes boundary conditions to the maximally isotropic subgroups of the Drinfeld double.

In Sect. 2, we review the general concept of the open string σ-models with particular emphasis on the global characterization of the D-branes boundary conditions. Then we review the Poisson–Lie T-duality in the presence of D-branes. In Sect. 3, we work out in detail the case of the $SL(2, \mathbf{C})/SU(2)$ WZW model. We consider the maximally isotropic subgroups of the Drinfeld double $SL(2, \mathbf{C})$, we derive from them the AdS_3 D-branes studied by Ponsot, Schomerus and Teschner and we describe the de Sitter duals of the AdS_3 D-branes. Finally, in Sect. 4, we explain the existence of their residual loop group symmetry.

2 Open Strings and D-Branes

2.1 Generalities

2.1.1. In the closed string case, the classical dynamics of nonlinear σ-model is completely characterized by a metric $d\sigma^2 = \frac{1}{2}G_{ij}dx^i dx^j$ and by a closed three-form H on a target manifold T. When $H = dB$ for some two-form $B = \frac{1}{2}B_{ij}dx^i \wedge dx^j$, the latter is called the B-field and the (Euclidean) action of the σ-model can be written as

$$S[x] = i \int d\bar{z} \wedge dz (G_{ij}(x) + B_{ij}) \partial_{\bar{z}} x^i \partial_z x^j.$$

But also when H is cohomologically nontrivial (i.e. there is no globally defined potential B) the classical model can be perfectly defined. Indeed, consider for definiteness the Riemann sphere as the closed string world-sheet. Then the (Euclidean) σ-model action can be cast in the WZW-like way:

$$S[x] = i \int_{\partial \Omega} d\bar{z} \wedge dz G_{ij}(x) \partial_{\bar{z}} x^i \partial_z x^j + i \int_{\Omega} \tilde{x}^* H. \tag{1}$$

Here Ω is a three-dimensional ball whose boundary $\partial\Omega$ is the Riemann sphere, the notation $\tilde{x}^*H$ means the pull-back of H to Ω via $\tilde{x}$, and the latter is the (extension) map from the ball Ω to the target T whose boundary value $\tilde{x}^i|_{\partial\Omega}$ is the σ-model configuration $x^i(z,\bar{z})$. Of course, the classical action (1) can be defined only when the topology of the target T permits to extend every map x on $\partial\Omega$ to the map $\tilde{x}$ on Ω. Moreover, this extension need not be unique. Two different extensions can give a different value of the action S. Although the classical dynamics (i.e. the field equations) is well defined and does not depend on this ambiguity, the path integral quantization is impossible unless the ambiguity of the action is $2\pi in$ with n an integer. If this is the case, the three-form H defines an integer-valued cocycle in the singular cohomology of T.

2.1.2. The first complete discussion (covering also the case of the non-exact H) of the open string case was given in [8]. The σ-model is again characterized by the target T, the metric $d\sigma^2$ and the closed three-form field H but also by a two-form α living on a submanifold P of T. Needless to say, the submanifold P is referred to as a D-brane. The boundary of the world-sheet has to lie in P and the restriction H_P of H on P must admit α as its potential, i.e. $d\alpha = H_P$.

It is instructive to write down the open string variational principle in the particular case where the world-sheet is the disc $|z| \leq 1$. We shall view this disc as the southern half of the Riemann sphere and denote it as $S_\downarrow$. We shall need to denote also the northern hemisphere as $S^\uparrow$. Let now $x^i(z,\bar{z})$ be a σ-model configuration defined on $S_\downarrow$ and having the boundary values $x^i(|z|=1)$ in the brane $P \subset T$. The action of this open string configuration is then given by

$$S[x] = i \int_{S_\downarrow} d\bar{z} \wedge dz\, G_{ij}(x)\partial_{\bar{z}}x^i \partial_z x^j + i \int_\Omega \tilde{x}^*H - i \int_{S^\uparrow} x_P^*\alpha. \tag{2}$$

Here x_P^i is an arbitrary map from $S^\uparrow$ to the brane P coinciding with the open string configuration $x^i(z,\bar{z})$ on the common boundary of $S^\uparrow$ and $S_\downarrow$. In other words: $x^i(|z|=1) = x_P^i(|z|=1)$. The (extension) map $\tilde{x}^i$ from Ω to T must now fulfill the following boundary conditions: $\tilde{x}^i|_{S_\downarrow} = x^i$ and $\tilde{x}^i|_{S^\uparrow} = x_P^i$.

We note that the open string action principle involves two extensions: first we extend the σ-model configuration $x^i(z,\bar{z})$ from the hemisphere $S_\downarrow$ to the whole sphere $\partial\Omega = S_\downarrow \cup S^\uparrow$ by choosing x_P^i on $S^\uparrow$ (in a sense we complete the sphere in the brane P) and then we extend $x^i \cup x_P^i$ to the map $\tilde{x}^i$. The existence and ambiguity of these extensions depend on the topology of T and P. The detailed discussion of this issue is given in [8]. Here we only note three things: (1) for all choices of T and P considered in this paper, the existence of the extensions will be guaranteed; (2) the existence of the extensions itself guarantees the unambiguous definition of the field equations of the model (essentially due to the property $d\alpha = H_P$); (3) the non-unicity of the extensions is an important issue for the *quantization*. Actually, the path integral quantization is impossible unless the ambiguity of the action is $2\pi in$ with n an integer. If this is the case, the pair (H, α) defines an integer-valued cocycle in the relative singular cohomology of T with respect to P [8].

The reader might not be accustomed with writing the non-metric part of the open string σ-model (2) as

$$i \int_\Omega \tilde{x}^*H - i \int_{S^\uparrow} x_P^*\alpha. \tag{3}$$

Usually the people write instead

$$i \int_{S_\downarrow} x^*B + i \int_{|z|=1} x^*A, \tag{4}$$

where B is the two-form potential B such that $dB = H$ and A is a one-form on the brane P. The advantage of the formulation (4) is clear: we do not need to consider any extension $\tilde{x}$ or x_P whatsoever. However, the formulation (4) is less general since it can be derived from (3) only when two requirements are satisfied: (1) the three-form H must be exact on the target T (i.e. $H = dB$ for some two-form B on T); (2) the (closed) two-form $(B_P - \alpha)$ must be exact on the brane P (i.e. $B_P - \alpha = dA$ for some one-form A on P). Of course, here B_P is the restriction of the form B on the brane P. It is immediate to verify that (3) yields (4) if these two conditions are satisfied.

It turns out that even in the cases when one can reduce the description (3) to (4), it is sometimes better to work with the invariant description (3). For example, the symmetry structure of the brane configuration is typically more transparent in the invariant formulation (3).

2.1.3. Consider now a closed string background $(T, d\sigma^2, H)$. As we already know, the open string generalization can be defined, if we add two more data (P, α). Of course, there are many possible quintuples (T, ds^2, H, P, α) to consider. We may wish to impose further restrictive conditions on these data, according to the aspects of the open string dynamics that we wish to study. There are two examples of such additional conditions:

 (i) In the framework of the WZW-like models, we wish that the part of the Kac–Moody symmetry be preserved by the boundary conditions. Such requirement typically ensures the preservation of the conformal symmetry of the open string model.
(ii) If the closed string model $(T, d\sigma^2, H)$ admits a T-dual $(T', d\sigma'^2, H')$, we require that also the open string model $(T, d\sigma^2, H, P, \alpha)$ admits a T-dual $(T', d\sigma'^2, H', P', \alpha')$.

Ponsot, Schomerus and Teschner [11] have looked for the D-brane boundary conditions satisfying the criterion (i). In distinction to it, we shall organize our paper from the point of view of the requirement (ii). In fact, if the bulk $(T, d\sigma^2, H)$-model is just the Poisson–Lie dualizable σ-model on the target D/G_1 (see Introduction), then there is a simple method [7, 8] to generate the D-brane shapes P and the boundary conditions α fulfilling the requirement (ii). In fact, take any element d of the corresponding Drinfeld double D and any maximally isotropic subgroup M of D. We shall see in a while that the pair (M, d) then completely determines the T-dualizable quintuple $(T, d\sigma^2, H, P, \alpha)$. In the case of the $SL(2, \mathbf{C})/SU(2)$ WZW model, there is another pleasing circumstance of this construction; namely, the obtained T-dualizable quintuple $(T, d\sigma^2, H, P, \alpha)$ preserves the residual M-loop group symmetry hence it automatically satisfies also the criterion (i). This observation explains, why we obtain the same brane configurations in the Euclidean AdS_3 as Ponsot, Schomerus and Teschner [11].

2.2 D-Branes and Poisson–Lie T-Duality

2.2.1. This subsection is a brief review of the results[2] [7–9]. It serves to keep complete the logical skeleton of the paper. However, the reader wishing to enter technical details must consult those papers. Consider the following first order action on the Drinfeld double D:

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[2]Some conventions in this paper are changed with respect to our recent article [9]. For example, the relation between the world-sheet coordinates σ, τ and $z, \bar{z}$, the normalization of the line element $d\sigma^2$ etc. We did those changes in order to have the same conventions as the paper [11] of Ponsot, Schomerus and Teschner. This will permit us the direct comparison of our results with theirs.

$$S = \frac{1}{2}\int_{\partial\Omega} d\sigma \wedge d\tau (\partial_\tau ll^{-1}, \partial_\sigma ll^{-1})_\mathcal{D} + \frac{1}{12}\int_\Omega (dll^{-1} \,\hat{,}\, [dll^{-1} \,\hat{,}\, dll^{-1}])_\mathcal{D}$$

$$- \frac{1}{2}\int_{\partial\Omega} d\sigma \wedge d\tau (\partial_\sigma ll^{-1}, \mathcal{E}\partial_\sigma ll^{-1})_\mathcal{D}. \tag{5}$$

Here $l(\sigma, \tau)$ is a map from the world-sheet into the double and the world-sheet coordinates are defined as

$$z = \tau + i\sigma, \qquad \bar{z} = \tau - i\sigma, \qquad \partial_z = \frac{1}{2}(\partial_\tau - i\partial_\sigma), \qquad \partial_{\bar{z}} = \frac{1}{2}(\partial_\tau + i\partial_\sigma).$$

The operator $\mathcal{E} : Lie(D) \to Lie(D)$ is self-adjoint with respect to the (metric) bilinear form $(., .)_\mathcal{D}$ on $\mathcal{D} = Lie(D)$ and it holds $\mathcal{E}^2 = \pm Id$. Actually, here we shall consider only the case $\mathcal{E}^2 = -Id$, because it leads to the T-duality between σ-models with real Euclidean actions [9].

For the moment, the action (5) is considered on a world-sheet without boundaries and it encodes the closed strings Poisson–Lie T-duality between the targets $T_1 = D/G_1$ and $T_2 = D/G_2$. Indeed, consider the coset D/G_j and parametrize[3] it by the elements f_j of D. With this parametrization of D/G_j, we may parametrize the surface $l(\tau, \sigma)$ in the double as follows

$$l(\tau, \sigma) = f_j(\tau, \sigma)g_j(\tau, \sigma), \quad g_j \in G_j \tag{6}$$

and there is no summing over j. The action (5) then becomes

$$S = \frac{1}{2}\int_{\partial\Omega} (f_j^{-1}\partial_\tau f_j, f_j^{-1}\partial_\sigma f_j)_\mathcal{D} + \frac{1}{12}\int_\Omega (df_j f_j^{-1} \,\hat{,}\, [df_j f_j^{-1} \,\hat{,}\, df_j f_j^{-1}])_\mathcal{D}$$

$$+ \int_{\partial\Omega} (\partial_\sigma g_j g_j^{-1}, f_j^{-1}\partial_\tau f_j)_\mathcal{D}$$

$$- \frac{1}{2}\int_{\partial\Omega} (f_j^{-1}\partial_\sigma f_j + \partial_\sigma g_j g_j^{-1}, \mathcal{E}_{f_j}(f_j^{-1}\partial_\sigma f_j + \partial_\sigma g_j g_j^{-1}))_\mathcal{D}, \tag{7}$$

where $\mathcal{E}_{f_j} = Ad_{f_j^{-1}}\mathcal{E}Ad_{f_j}$ and we tacitly suppose the measure $d\sigma \wedge d\tau$ present in the formula. Now we note that the expression (7) is Gaussian in the $Lie(G_j)$-valued variable $\partial_\sigma g_j g_j^{-1}$. The most useful strategy to solve it away is to pick up some basis S_j^a in $Lie(G_j)$, write $\partial_\sigma g_j g_j^{-1} = \mu_{ja}S_j^a$ and integrate away μ_{ja}. This gives

$$S = \frac{1}{2}\int_{\partial\Omega} d\bar{z} \wedge dz (\partial_{\bar{z}} f_j f_j^{-1}, \partial_z f_j f_j^{-1})_\mathcal{D}$$

$$+ \frac{1}{12}\int_\Omega (df_j f_j^{-1} \,\hat{,}\, [df_j f_j^{-1} \,\hat{,}\, df_j f_j^{-1}])_\mathcal{D}$$

$$+ i\int_{\partial\Omega} d\bar{z} \wedge dz (f_j^{-1}\partial_{\bar{z}} f_j, S_j^a + i\mathcal{E}_{f_j}S_j^a)_\mathcal{D}(A_{f_j}^{-1})_{ab}(S_j^b, f_j^{-1}\partial_z f_j)_\mathcal{D}, \tag{8}$$

where

$$A_{f_j}^{ab} = (S_j^a, \mathcal{E}_{f_j}S_j^b)_\mathcal{D}. \tag{9}$$

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[3]If there exists no global section of this fibration, we can choose several local sections covering the whole base space D/G_j.

We note that in spite of the explicit presence of the imaginary unit in this formula, the σ-model action (8) is always real. The duality is the equivalence of the models (8) for different j.

By the way, the σ-model like (8) can be associated to every maximally isotropic subgroup of $\mathcal{D}$ provided the corresponding matrix A_f is invertible. The target of such a σ-model is the coset of the Drinfeld double by this subgroup.

2.2.2. We wish to consider the first order action (5) on the hemisphere (or disc) $S_\downarrow$. For this, we choose some maximally isotropic[4] subgroup M of D and an element $d \in D$ and require that $l(z, \bar{z}) \in Md$ for $|z| = 1$. The submanifold Md is the left M-orbit of the element d and we shall call it a first order D-brane. The first and the third terms of the action (8) can be perfectly defined as integrals over $S_\downarrow$ only. As it is discussed in [8] and in Sect. 2.1, giving sense to the second term requires to choose a two-form α_D on the first order brane Md such that $d\alpha_D$ is the restriction of the three-form $1/6(dll^{-1} \stackrel{\wedge}{,} [dll^{-1} \stackrel{\wedge}{,} dll^{-1}])_\mathcal{D}$ on Md. In the spirit of Sect. 2.1, the open string first order action then reads:

$$S = \frac{1}{2} \int_{S_\downarrow} d\sigma \wedge d\tau (\partial_\tau ll^{-1}, \partial_\sigma ll^{-1})_\mathcal{D} - \frac{1}{2} \int_{S_\downarrow} d\sigma \wedge d\tau (\partial_\sigma ll^{-1}, \mathcal{E}\partial_\sigma ll^{-1})_\mathcal{D}$$
$$+ \frac{1}{12} \int_\Omega (dll^{-1} \stackrel{\wedge}{,} [dll^{-1} \stackrel{\wedge}{,} dll^{-1}])_\mathcal{D} - \frac{1}{2} \int_{S^\uparrow} l^* \alpha_D. \tag{10}$$

We ask the reader to excuse us some abuse of notation: we should have denoted by l the true $S_\downarrow$-open string configuration, by l_{Md} its $S^\uparrow$ extension in the first order brane Md and by $\tilde{l}$ the extension of both l and l_{Md} to the ball Ω. Instead, we have used everywhere l and hoped not to cause a confusion.

Now remark, that the D-invariance of the metric $(.,.)_\mathcal{D}$ means that not only M but also Md is the isotropic surface in the double D. This fact implies that the restriction of $1/6(dll^{-1} \stackrel{\wedge}{,} [dll^{-1} \stackrel{\wedge}{,} dll^{-1}])_\mathcal{D}$ on Md simply vanishes. Thus we fully define the open string model (10) by choosing a closed form α_D on Md. We make the simplest possible choice and set

$$\alpha_D = 0.$$

Let G_1 and G_2 be two maximally isotropic subgroups of D. If the respective matrices A_f given by (9) are invertible for both choices of G_1 and G_2 we know that there is the bulk T-duality between the model (8) on D/G_1 and its counterpart on D/G_2. This duality takes place also in the open string case if we start with the action (10), the boundary condition $l(|z| = 1) \in Md$ and $\alpha_D = 0$.

We shall not repeat here the derivation of the mutually dual quintuples $(T_j, d\sigma_j^2, H_j, P_j, \alpha_j)$, $j = 1, 2$ obtained from the first-order action (10). It has been done in [8] at it is based on the following variant of the Polyakov–Wiegmann formula [12]:

$$(fg)^* WZW(l) = f^* WZW(l) + g^* WZW(l) - d(f^*(l^{-1}dl) \stackrel{\wedge}{,} g^*(dll^{-1}))_\mathcal{D}. \tag{11}$$

Here

$$WZW(l) \equiv 1/6(dll^{-1} \stackrel{\wedge}{,} [dll^{-1} \stackrel{\wedge}{,} dll^{-1}])_\mathcal{D}$$

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[4]This means that the $Lie(M)$ is the maximally isotropic subspace of $Lie(D)$ with respect to the canonical indefinite metric $(.,.)_\mathcal{D}$.

and the two maps from D to D: $f(l) = f$ and $g(l) = g$ are induced by the decomposition (6). As usual, $*$ denotes the pull-back of the forms under the mappings to the group manifold D. The result of the derivation is as follows:

(1) The manifold T_j is the coset D/G_j.

(2–3) The metric $d\sigma_j^2$ and the H_j-field are given by the bulk σ-model (8). Explicitly:

$$ds_j^2 = \frac{1}{2}(f_j^{-1}df_j, S_j^a)_{\mathcal{D}}(A_{f_j}^{-1})_{ab}(S_j^b, f_j^{-1}df_j)_{\mathcal{D}}; \tag{12}$$

$$H_j = \frac{i}{2}d\{(f_j^{-1}df_j, \mathcal{E}_{f_j}S_j^a)_{\mathcal{D}} \wedge (A_{f_j}^{-1})_{ab}(S_j^b, f_j^{-1}df_j)_{\mathcal{D}}\}$$
$$- \frac{i}{12}(df_j f_j^{-1} \stackrel{\wedge}{,} [df_j f_j^{-1} \stackrel{\wedge}{,} df_j f_j^{-1}])_{\mathcal{D}}.$$

(4) The D-brane P_j in D/G_j is the image of $Md \subset D$ under the coset projection $\pi_{G_j} : D \rightarrow D/G_j$.

(5) The form α_j on P_j is given by

$$\alpha_j = \frac{i}{2}(f_j^{-1}df_j, \mathcal{E}_{f_j}S_j^a)_{\mathcal{D}} \wedge (A_{f_j}^{-1})_{ab}(S_j^b, f_j^{-1}df_j)_{\mathcal{D}}$$
$$- \frac{i}{2}(f_j^{-1}df_j \stackrel{\wedge}{,} dg_j(f_j)g_j(f_j)^{-1})_{\mathcal{D}}. \tag{13}$$

Recall that $f_j \in D$ are the representatives of the coset elements in D/G_j and $g_j(f_j)$ is any map from P_j to G_j, such that $f_jg_j(f_j) \in Md$. The ambiguity in the definition of the map $g_j : P_j \rightarrow G_j$ does not influence the form α_j. Moreover, the maps g_j may exist only locally on P_j. However, the form α_j is defined *globally* on P_j and has to be glued from the maps g_j defined on local charts forming a covering of P_j.

3 D-Branes in the Euclidean AdS_3

3.1 The Bulk Story

Here we apply the general results of Sect. 2 to the $SL(2, \mathbf{C})/SU(2)$ WZW model. The Drinfeld double D is the group $SL(2, \mathbf{C})$ viewed as a real group and the biinvariant maximally Lorentzian metric on it is naturally induced from the following non-degenerate invariant symmetric bilinear form $(.,.)_{\mathcal{D}}$ on its Lie algebra $\mathcal{D} = sl(2, \mathbf{C})$:

$$(x, y)_{\mathcal{D}} = \operatorname{Im} \operatorname{Tr}(xy).$$

In other words, the indefinite metric is given by the imaginary part of the trace in the fundamental representation of $sl(2, \mathbf{C})$. For the maximally isotropic subgroups G_1 and G_2 we take, respectively, $SU(2)$ and $SL^a(2, \mathbf{R})$. The superscript a means that the group $SL(2, \mathbf{R})$ is atypically embedded into $SL(2, \mathbf{C})$ according to the following formula [9]:

$$\begin{pmatrix} \mu & i\nu \\ i\rho & \lambda \end{pmatrix} \in SL^a(2, R), \quad \mu, \nu, \rho, \lambda \in \mathbf{R}, \ \mu\lambda + \nu\rho = \mathbf{1}.$$

In fact, our embedding $SL^a(2, \mathbf{R})$ is conjugated to the standard one (real matrices with unit determinant) by the following element of $SL(2, \mathbf{C})$:

$$\frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ i-1 & 1-i \end{pmatrix}.$$

We recall that σ-models (8) on D/G and on D/G' have the same target geometry if G and G' are conjugated in D. This means that we could equally well consider the standardly embedded $SL(2, \mathbf{R})$ as the maximally isotropic subgroup G_2. Our atypical choice $G_2 = SL^a(2, \mathbf{R})$ is motivated by an effort to make as straightforward as possible the comparison of our results with those of Ponsot, Schomerus and Teschner [11]. We also note that the isotropy of the Lie algebras $su(2)$ and $sl^a(2, \mathbf{R})$ is clear because they are both the *real* forms of $SL(2, \mathbf{C})$.

Finally, we need the operator $\mathcal{E}$. It is simply the multiplication by the imaginary unit in $sl(2, \mathbf{C})$ viewed as the *real* Lie algebra. Now we can write the first order action (10) in this particular case:

$$S = \frac{1}{2}\int_{\partial\Omega} d\sigma \wedge d\tau \operatorname{Im}\operatorname{Tr}(\partial_\tau ll^{-1}\partial_\sigma ll^{-1}) + \frac{1}{12}\int_\Omega \operatorname{Im}\operatorname{Tr}(dll^{-1}\wedge[dll^{-1}\stackrel{\wedge}{,}dll^{-1}])$$
$$- \frac{1}{2}\int_{\partial\Omega} d\sigma \wedge d\tau \operatorname{Im}\operatorname{Tr}(\partial_\sigma ll^{-1}i\partial_\sigma ll^{-1}). \tag{14}$$

This expression can be cast even more simply as

$$S = \int_{\partial\Omega} d\sigma \wedge dz \operatorname{Im}\operatorname{Tr}(\partial_z ll^{-1}\partial_\sigma ll^{-1}) + \frac{1}{12}\int_\Omega \operatorname{Im}\operatorname{Tr}(dll^{-1}\wedge[dll^{-1}\stackrel{\wedge}{,}dll^{-1}]).$$

We recall that the T-duality relates the σ-models living on $SL(2, \mathbf{C})/SU(2)$ and on $SL(2, \mathbf{C})/SL^a(2, \mathbf{R})$. Since we already know the general formulae (12) and (13), we need only to decompose the world-sheet $l(\sigma, \tau)$ in the double $D = SL(2, \mathbf{C})$ as $l = f_j g_j, j = 1, 2$ (cf. (6)). The section f_2 was constructed in [9]:

$$f_2 = \begin{pmatrix} \cos\vartheta + i\frac{L}{\sqrt{L^2+1}}\sin\vartheta & i\frac{1}{\sqrt{L^2+1}}\sin\vartheta \\ i\frac{1}{\sqrt{L^2+1}}\sin\vartheta & \cos\vartheta - i\frac{L}{\sqrt{L^2+1}}\sin\vartheta \end{pmatrix}$$
$$\times \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}, \tag{15}$$

where $0 \leq \vartheta \leq \pi, 0 \leq \chi \leq \pi/2$ and $L \in \mathbf{R}$.

For the case $j = 1$, we parametrize the coset $SL(2, \mathbf{C})/SU(2)$ differently as in [9]. For the sake of being notationally as close as possible to the paper [11], we take for f_1 the following section

$$f_1 = \begin{pmatrix} e^{\phi/2} & 0 \\ \gamma e^{\phi/2} & e^{-\phi/2} \end{pmatrix}, \tag{16}$$

where $\phi \in \mathbf{R}$ and $\gamma \in \mathbf{C}$. Now we can directly calculate from (12) the σ-model metrics and H-fields for the both cases $j = 1, 2$:

$$d\sigma_1^2 = -\frac{1}{4}(d\phi^2 + e^{2\phi}d\gamma d\bar\gamma), \qquad H_1 = -\frac{1}{2}e^{2\phi}d\phi \wedge d\bar\gamma \wedge d\gamma; \tag{17}$$

$$d\sigma_2^2 = -\frac{1}{4}\frac{(dL)^2}{L^2+1}\sin^2 2\chi \sin^2 2\vartheta + (L^2+1)(d\chi^2 + \sin^2 2\chi d\vartheta^2)$$

$$+ \cos 2\vartheta dL d\chi - \frac{1}{2}\sin 4\chi \sin 2\vartheta dL d\vartheta;$$

$$H_2 = -4i\sqrt{L^2+1}\sin 2\chi dL \wedge d\chi \wedge d\vartheta. \tag{18}$$

$d\sigma_1^2$ and H_1 are equal to the metric and H-field of the $SL(2,\mathbf{C})/SU(2)$ WZW model [10, 11]. The metric $d\sigma_2^2$ turns out to be the de Sitter metric written in appropriate coordinates (cf. [9]). $d\sigma_2^2$ and H_2 together define the de Sitter σ-model introduced in [9].

It is very convenient to introduce the following Hermitian matrices

$$h = f_1 f_1^\dagger, \qquad s = f_2 \sigma_1 f_2^\dagger, \tag{19}$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the Pauli matrix and $\dagger$ means the Hermitian conjugation of matrices. In terms of h and s the σ-model backgrounds (17) and (18), can be respectively rewritten as

$$d\sigma_1^2 = -\frac{1}{8}\text{Tr}(dhh^{-1}dhh^{-1}), \tag{20a}$$

$$H_1 = -\frac{1}{24}\text{Tr}(dhh^{-1} \wedge [dhh^{-1} \overset{\wedge}{,} dhh^{-1}]), \tag{20b}$$

$$d\sigma_2^2 = -\frac{1}{8}\text{Tr}(dss^{-1}dss^{-1}), \tag{21a}$$

$$H_2 = -\frac{1}{24}\text{Tr}(dss^{-1} \wedge [dss^{-1} \overset{\wedge}{,} dss^{-1}]). \tag{21b}$$

In other words, the closed string $SL(2,\mathbf{C})/SU(2)$ WZW action can be written as

$$S = -\frac{i}{4}\int_{\partial\Omega} d\bar{z} \wedge dz\, \text{Tr}(\partial_{\bar{z}}hh^{-1}\partial_z hh^{-1}) - \frac{i}{24}\int_\Omega \text{Tr}(d\tilde{h}\tilde{h}^{-1} \wedge [d\tilde{h}\tilde{h}^{-1} \overset{\wedge}{,} d\tilde{h}\tilde{h}^{-1}]) \tag{22}$$

and the closed string de Sitter action as

$$S = -\frac{i}{4}\int_{\partial\Omega} d\bar{z} \wedge dz\, \text{Tr}(\partial_{\bar{z}}ss^{-1}\partial_z ss^{-1}) - \frac{i}{24}\int_\Omega \text{Tr}(d\tilde{s}\tilde{s}^{-1} \wedge [d\tilde{s}\tilde{s}^{-1} \overset{\wedge}{,} d\tilde{s}\tilde{s}^{-1}]). \tag{23}$$

Note that in (20–23) there is the full trace, not only its imaginary part. The original and the dual model look pretty the same but we must realize that the determinant of s is (-1) and that of h is 1. Moreover, the trace of h is positive.

The T-duality for the open strings depends on the choice of the first order D-branes Md. We shall separately consider three cases: (1) $M = SL^a(2,\mathbf{R})$; (2) $M = SU(2)$; (3) $M = AN$.

3.2 AdS_2 Branes

We start with the case $M = SL^a(2,\mathbf{R})$. The possible first order D-branes are then the sub-manifolds Md of $SL(2,\mathbf{C})$ with d being a fixed element of $SL(2,\mathbf{C})$. We do not make here the most general choice of d; instead, we consider only d having form

$$d = \begin{pmatrix} 1 & 0 \\ \frac{1}{2}c & 1 \end{pmatrix}, \quad c \in \mathbf{R}.$$

The motivation for this particular choice is simple: when we switch from the first order description (10) to the second order one (22), it reproduces the AdS_2 branes considered in [11].

Thus our first order D-branes are three-dimensional submanifolds of $SL(2, \mathbf{C})$ of the form

$$p = Md = SL^a(2, \mathbf{R})d = \begin{pmatrix} \mu & iv \\ i\rho & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{2}c & 1 \end{pmatrix} = \begin{pmatrix} \mu + \frac{1}{2}ivc & iv \\ i\rho + \frac{1}{2}\lambda c & \lambda \end{pmatrix}, \qquad (24)$$

where the real number c is fixed and $\mu, v, \rho, \lambda \in \mathbf{R}$ vary while respecting the constraint $\mu\lambda + v\rho = 1$.

3.2.1 The Target $D/G_1 = SL(2, \mathbf{C})/SU(2)$

Now we want to find the shape of the corresponding D-brane P_1 in the target $D/G_1 = SL(2, \mathbf{C})/SU(2)$. As it is well-known, the coset $SL(2, \mathbf{C})/SU(2)$ can be identified with the group AN, whose elements are the matrices of the form (16). There is another natural way to view this coset, namely, as a set of Hermitean matrices in $SL(2, \mathbf{C})$ with a positive trace and unit determinant. They can be parametrized as (cf. [11])

$$h = f_1 f_1^\dagger = \begin{pmatrix} e^\phi & e^\phi \bar{\gamma} \\ e^\phi \gamma & e^\phi \gamma \bar{\gamma} + e^{-\phi} \end{pmatrix}), \quad \phi \in \mathbf{R}, \ \gamma \in \mathbf{C}. \qquad (25)$$

The canonical coset projection map $h : SL(2, \mathbf{C}) \to SL(2, \mathbf{C})/SU(2)$ is simply given by

$$h(l) = ll^\dagger, \quad l \in SL(2, \mathbf{C}). \qquad (26)$$

It turns out that the restriction of the map h to the subgroup AN gives a diffeomorphism between the subgroup AN of $SL(2, \mathbf{C})$ and the space (25). Now we are ready to find the shape of the D-brane P_1 in D/G_1. It is given by the coset projection of the first order D-brane (24), or, in other words, by the Hermitian matrices of the form

$$h(p) = \begin{pmatrix} v^2(1 + \frac{1}{4}c^2) + \mu^2 & -i\mu\rho + iv\lambda(1 + \frac{1}{4}c^2) + \frac{1}{2}c \\ +i\mu\rho - iv\lambda(1 + \frac{1}{4}c^2) + \frac{1}{2}c & \lambda^2(1 + \frac{1}{4}c^2) + \rho^2 \end{pmatrix}.$$

Comparing with the parametrization (25), we observe that the D-brane P_1 in the coset $SL(2, \mathbf{C})/SU(2)$ is characterized by the equation

$$e^\phi(\gamma + \bar{\gamma}) = c \qquad (27)$$

with a constant c. This is exactly the AdS_2 brane considered in [11].

As we have said, the first order action (10) together with the choice of the first order D-brane (24) (and $\alpha_D = 0$) determines the open string quintuple $(T_1, ds_1^2, H_1, P_1, \alpha_1)$. So far we have determined four its elements: the target $T_1 = SL(2, \mathbf{C})/SU(2)$ is parametrized as in (25), then

$$ds_1^2 = -\frac{1}{4}(d\phi^2 + e^{2\phi} d\bar{\gamma} d\gamma), \qquad H_1 = -\frac{1}{2}e^{2\phi} d\phi \wedge d\bar{\gamma} \wedge d\gamma \qquad (17)$$

and the D-brane P_1 is characterized by (27). It remains to determine α_1. We use the formula (13) and argue that the second term on its r.h.s. vanishes in this particular case. Indeed, we can choose the map g_1 equal to $g_1(f_1) = 1$. Hence we conclude

$$\alpha_1 = \frac{1}{4}e^{2\phi} d\gamma \wedge d\bar{\gamma}. \qquad (28)$$

It is important to note that the variables ϕ and γ in (28) are subject to the constraint (27).

Having obtained the quintuple $(T_1, d\sigma_1^2, H_1, P_1, \alpha_1)$, we can write down the action for open strings. In the spirit of the discussion in Sect. 2.1, we pick up some globally defined two-form potential B_1 such that $dB_1 = H_1$. We know that the combination $B_1 - \alpha_1$ on the D-brane is always a closed form. In the present case it is even exact therefore it has a globally defined potential A_1 on P_1. The non-metric part of the σ-model action (22) on the upper half-plane can now be written as follows

$$i \int_{S_\downarrow} B_1 + i \int_{|z|=1} A_1.$$

Actually, in our situation, we can take

$$B_1 = \frac{1}{4} e^{2\phi} d\gamma \wedge d\bar{\gamma}.$$

Clearly, the difference $B_1 - \alpha_1$ on the D-brane P_1 vanishes hence $A_1 = 0$. In other words, we can cast the open string action without any boundary term as follows

$$S = -\frac{i}{2} \int_{S_\downarrow} d\bar{z} \wedge dz (\partial_z \phi \partial_z \phi + e^{2\phi} \partial_z \gamma \partial_{\bar{z}} \bar{\gamma}). \tag{29}$$

We stress again that this is the *open* string action. Nevertheless the integration over the boundary $|z| = 1$ is missing due to the clever choice of the potential B_1. The boundary conditions that accompany the action (29) say that the boundary of the world-sheet lies in the D-brane P_1.

We could be happy that in the particular case of AdS_2 branes in the Euclidean AdS_3 the open string action can be written without any extensions of the type x_p and $\tilde{x}$ (cf. (2)). On the other hand, the action (29) is expressed in the coordinates ϕ, γ and this fact makes technically complicated to verify whether there is a residual loop group symmetry. Indeed, the loop group action on the configuration ϕ, γ is given by a cumbersome formula and it would be tedious to check the residual symmetry by working directly with (29). It turns out that it is much easier to treat the symmetry issues in the $(\int_\Omega \tilde{x}^* H - \int_{S\uparrow} x_P^* \alpha)$-representation of the non-metric part of the open string action, because it can be done without the necessity of introducing any coordinates on the target $SL(2, \mathbf{C})/SU(2)$. Let us see how this works.

The points in the coset $T_1 = SL(2, \mathbf{C})/SU(2)$ are Hermitian $2d$-matrices h with unit determinant and positive trace. We wish to express the data $d\sigma_1^2, H_1, P_1, \alpha_1$ in terms of h. Here is the answer

$$d\sigma_1^2 = -\frac{1}{8} \operatorname{Tr}(dh h^{-1} dh h^{-1}); \tag{20a}$$

$$H_1 = -\frac{1}{24} \operatorname{Tr}(dh h^{-1} \wedge [dh h^{-1} \,\hat{,}\, dh h^{-1}]); \tag{20b}$$

$$P_1 = \{h; \operatorname{Tr}(h\sigma_1) = c\}; \tag{30a}$$

$$\alpha_1 = +\frac{c}{16 + 4c^2} \operatorname{Tr}(\sigma_1 h \sigma_1 dh \wedge \sigma_1 dh), \quad h \in P_1. \tag{30b}$$

Here we recall that $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the Pauli matrix. The reader can easily verify, that inserting the parametrization (25) into the formulae (20a–20b) and (30a–30b), gives respectively formulae (17) and (27), (28).

Thus we conclude the discussion of the AdS_2 branes in the $SL(2, \mathbf{C})/SU(2)$ background by writing for them the open string action in the coordinate free way:

$$
\begin{aligned}
S = &-\frac{i}{4} \int_{S_\downarrow} d\bar{z} \wedge dz \, \mathrm{Tr}(\partial_{\bar{z}} h h^{-1} \partial_z h h^{-1}) \\
&- \frac{i}{24} \int_\Omega \mathrm{Tr}(d\tilde{h}\tilde{h}^{-1} \wedge [\tilde{h}\tilde{h}^{-1} \,\hat{,}\, d\tilde{h}\tilde{h}^{-1}]) \\
&- i \frac{c}{16 + 4c^2} \int_{S^\uparrow} \mathrm{Tr}(\sigma_1 h_P \sigma_1 dh_P \wedge \sigma_1 dh_P).
\end{aligned}
\tag{31}
$$

Recall once more that h_P (lying in the brane P_1) is the $S^\uparrow$-extension of (the open string configuration) h and $\tilde{h}$ is the extension of $h \cup h_P$ to the interior of the ball Ω. We shall show in Sect. 4, that the action (31) possesses a residual $SL^a(2, \mathbf{R})$ loop group symmetry.

3.2.2 The Target $D/G_2 = SL(2, \mathbf{C})/SL^a(2, \mathbf{R})$

In order to find the shape of the dual D-branes, it is convenient to switch from the parametrization (15) of the coset $SL(2, \mathbf{C})/SL^a(2, \mathbf{R})$ to the parametrization by the Hermitian $2d$-matrices with determinant equal to (-1). They can be written as

$$
s = \begin{pmatrix} u & w \\ \bar{w} & v \end{pmatrix}, \quad uv - \bar{w}w = -1.
\tag{32}
$$

Clearly, $SL(2, \mathbf{C})/SL^a(2, \mathbf{R})$ is nothing but the de Sitter space, if it is equipped with the Minkowski metric

$$
ds_{dS}^2 = dudv - d\bar{w}dw
$$

restricted to the surface $uv - \bar{w}w = -1$. Two parametrizations are related by $s = f_2 \sigma_1 f_2^\dagger$ (cf. (15)), that gives

$$
\frac{1}{2}(u + v) = L, \qquad \frac{1}{2}(u - v) = L \cos 2\chi + \sin 2\chi \cos 2\vartheta,
\tag{33a}
$$

$$
w = \cos 2\chi - L \sin 2\chi \cos 2\vartheta - i\sqrt{L^2 + 1} \sin 2\chi \sin 2\vartheta.
\tag{33b}
$$

The reader may verify that, indeed, it holds $uv - \bar{w}w = -1$.

The coset projection $s : SL(2, \mathbf{C}) \to SL(2, \mathbf{C})/SL^a(2, \mathbf{R})$ is then given by

$$
s(l) = l\sigma_1 l^\dagger.
\tag{34}
$$

The shape of the dual de Sitter D-brane is now given by the projection of the first-order D-brane (24):

$$
s(p) = \begin{pmatrix} v^2 c & 1 + iv\lambda c \\ 1 - iv\lambda c & \lambda^2 c \end{pmatrix}.
$$

If $c = 0$, then the dual D-brane is just a point $w = 1$, $u = v = 0$. If $c \neq 0$, then the dual D-brane P_2 is two-dimensional and is characterized by the following relations

$$\mathrm{Re}(w) = 1, \qquad \mathrm{sign}(u) = \mathrm{sign}(v) = \mathrm{sign}(c). \tag{35}$$

Thus the dual AdS_2 brane has a conical shape. The remaining missing ingredient of the open string quintuple is the form α_2 on P_2. It is obtained from (13) but the straightforward calculation based on the coset parametrization (15) would be exceedingly tedious. Instead, we shall use the fact that the form α_2 is defined only on the brane P_2. In this case, we can parametrize the section f_2 as follows

$$f_{P_2} = \begin{pmatrix} 1 + \frac{i}{2}\sqrt{uv} & \frac{1}{2}u \\ \frac{1}{2}v & 1 - \frac{i}{2}\sqrt{uv} \end{pmatrix}, \tag{36}$$

since it is easy to see that the coset projection map s gives

$$s(f_{P_2}) = f_{P_2}\sigma_1 f_{P_2}^\dagger = \begin{pmatrix} u & 1 + i\sqrt{uv} \\ 1 - i\sqrt{uv} & v \end{pmatrix}. \tag{37}$$

This is indeed the D-brane surface P_2 because the matrix on the r.h.s. of (37) is the general solution of the relations (35).

Using the parametrization (36), it is now easy to calculate the form α_2 from the formula (13). The result is simple:

$$\alpha_2 = \frac{i}{4} \frac{du \wedge dv}{\sqrt{uv}}. \tag{38}$$

Of course, for $c = 0$ the brane reduces to a point and the form α_2 automatically vanishes.

Thus we conclude the discussion of the de Sitter dual of the AdS_2 branes by writing for them the open string action in the global way (3):

$$S = -\frac{i}{4} \int_{S_\downarrow} d\bar{z} \wedge dz \, \mathrm{Tr}(\partial_{\bar{z}} s s^{-1} \partial_z s s^{-1})$$

$$- \frac{i}{24} \int_\Omega \mathrm{Tr}(d\tilde{s}\tilde{s}^{-1} \wedge [d\tilde{s}\tilde{s}^{-1} \stackrel{\wedge}{,} \partial\tilde{s}\tilde{s}^{-1}]) + \frac{1}{4} \int_{S^\uparrow} \frac{du_P \wedge dv_P}{\sqrt{u_P v_P}}. \tag{39}$$

Recall once more that

$$s_P = \begin{pmatrix} u_P & 1 + i\sqrt{u_P v_P} \\ 1 - i\sqrt{u_P v_P} & v_P \end{pmatrix}$$

is the $S^\uparrow$-extension of s to the brane P_2 and $\tilde{s}$ is the extension of $s \cup s_P$ to the interior of the ball Ω. We shall show in Sect. 4, that the action (39) possesses a residual $SL^a(2, \mathbf{R})$ loop group symmetry.

3.3 Spherical Branes

We continue with the case $M = SU(2)$ embedded into $SL(2, \mathbf{C})$ in the standard way:

$$\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \quad \alpha, \beta \in \mathbf{C}, \ \alpha\bar{\alpha} + \beta\bar{\beta} = 1.$$

The possible first order D-branes are then the submanifolds Md of $SL(2, \mathbf{C})$ with d being a fixed element of $SL(2, \mathbf{C})$. As in the previous case of the AdS_2 branes, we do not make here the most general choice of d; instead, we consider only d having form

$$d = \begin{pmatrix} e^\rho & 0 \\ 0 & e^{-\rho} \end{pmatrix}, \quad \rho \in \mathbf{R}.$$

The motivation for this particular choice is simple: if we switch from the first order action (10) to the second order one (22), it reproduces the *spherical* branes considered in [11].

Thus our first order D-branes are three-dimensional submanifolds of $SL(2, \mathbf{C})$ of the form

$$p = \begin{pmatrix} e^\rho \alpha & -e^{-\rho} \bar\beta \\ e^\rho \beta & e^{-\rho} \bar\alpha \end{pmatrix}, \tag{40}$$

where the real number ρ is fixed and $\alpha, \beta \in \mathbf{C}$ vary while respecting the constraint $\alpha\bar\alpha + \beta\bar\beta = 1$.

3.3.1 The Target $D/G_1 = SL(2, \mathbf{C})/SU(2)$

Now we want to find the shape of the corresponding D-brane P_1 in the target $D/G_1 = SL(2, \mathbf{C})/SU(2)$. It is given by the coset projection (26) of the first order D-brane (40):

$$h(p) = \begin{pmatrix} \cosh 2\rho - \sinh 2\rho (\alpha\bar\alpha - \beta\bar\beta) & -2\alpha\bar\beta \sinh 2\rho \\ -2\bar\alpha\beta \sinh 2\rho & \cosh 2\rho + \sinh 2\rho (\alpha\bar\alpha - \beta\bar\beta) \end{pmatrix}.$$

Note that the trace of $h(p)$ is constant. Comparing with the parametrization (25), we thus observe that the D-brane P in the coset $SL(2, \mathbf{C})/SU(2)$ is characterized by the equation

$$\mathrm{Tr}(h) = e^\phi (1 + \bar\gamma\gamma) + e^{-\phi} = 2\cosh 2\rho \tag{41}$$

with constant ρ. This is exactly the spherical brane considered in [11].

As we have said, the first order action (10) together with the choice of the first order D-brane (40) (and $\alpha_D = 0$) determines the open string quintuple $(T_1, d\sigma_1^2, H_1, P_1, \alpha_1)$. So far we have determined its four elements: the target $T_1 = SL(2, \mathbf{C})/SU(2)$ is parametrized as in (25), then

$$ds_1^2 = -\frac{1}{8}\mathrm{Tr}(dhh^{-1}dhh^{-1}) = -\frac{1}{4}(d\phi^2 + e^{2\phi}d\bar\gamma d\gamma),$$

$$H_1 = -\frac{1}{24}\mathrm{Tr}(dhh^{-1} \wedge [dhh^{-1} \overset{\wedge}{,} dhh^{-1}]) = -\frac{1}{2}e^{2\phi}d\phi \wedge d\bar\gamma \wedge d\gamma$$

and the spherical D-brane P_1 is characterized by (41).

It remains to determine α_1. Of course, we use the general formula (13). The calculation is a bit tedious but straightforward and it gives a remarkably simple result:

$$\alpha_1 = -\frac{\cosh 2\rho}{8\sinh^2 2\rho}\mathrm{Tr}(hdh \wedge dh), \quad \mathrm{Tr}(h) = 2\cosh 2\rho. \tag{42}$$

It is crucial to stress that the form α_1 in the formula (42) is defined only on the surface (41).

Having obtained the quintuple $(T_1, ds_1^2, H_1, P_1, \alpha_1)$, we can write down the action for open strings attached to the spherical brane. The result is clearly

$$
S = -\frac{i}{4} \int_{S_\downarrow} d\bar{z} \wedge dz \, \mathrm{Tr}(\partial_{\bar{z}} h h^{-1} \partial_z h h^{-1})
$$

$$
- \frac{i}{24} \int_\Omega \mathrm{Tr}(d\tilde{h}\tilde{h}^{-1} \wedge [\tilde{h}\tilde{h}^{-1} \,\hat{,}\, d\tilde{h}\tilde{h}^{-1}])
$$

$$
+ i\frac{\cosh 2\rho}{8\sinh^2 2\rho} \int_{S^\uparrow} \mathrm{Tr}(h_P dh_P \wedge dh_P). \tag{43}
$$

Recall once more that h_P is the $S^\uparrow$-extension of h lying in the brane P_1 and $\tilde{h}$ is the extension of $h \cup h_P$ to the interior of the ball Ω. The boundary conditions that accompany the action (43) say that the boundary of the world-sheet lies in the D-brane (41). We shall show in Sect. 4, that the action (43) possesses a residual $SU(2)$ loop group symmetry.

3.3.2 The Target $D/G_2 = SL(2, \mathbf{C})/SL^a(2, \mathbf{R})$

It is straightforward to find the shapes of the de Sitter duals of the spherical D-branes. We just apply the coset projection map (34) on the first order brane (40) and obtain

$$
s(p) = \begin{pmatrix} -\alpha\beta - \bar{\alpha}\bar{\beta} & \alpha^2 - \bar{\beta}^2 \\ \bar{\alpha}^2 - \beta^2 & +\alpha\beta + \bar{\alpha}\bar{\beta} \end{pmatrix}.
$$

Note that the resulting brane P_2 can be characterized in the coordinate free way as

$$
P_2 = \{s; \mathrm{Tr}(s) = 0\}.
$$

In the parametrization (32), this means $u + v = 0$. Remembering that $\det s = -1$ this gives $u^2 + w\bar{w} = 1$ which is the equation of the sphere. Thus the dual D-branes of the spherical branes are also spherical.

The form α_2 is given by the formula (13). We work in the coordinates χ, ϑ, L in which the D-brane P_2 is characterized by the equation $L = 0$. It turns out that for $g_2(f_2)$ appearing in (13) we can take a constant map

$$
g_2(f_2) = \begin{pmatrix} e^\rho & 0 \\ 0 & e^{-\rho} \end{pmatrix},
$$

hence the second term in (13) vanishes. A direct calculation then shows that, on the surface $L = 0$, the first term also vanishes. Thus we obtain

$$
\alpha_2 = 0.
$$

We conclude the discussion of the de Sitter dual of the spherical branes by writing for them the open string action:

$$
S = -\frac{i}{4} \int_{S_\downarrow} d\bar{z} \wedge dz \, \mathrm{Tr}(\partial_{\bar{z}} s s^{-1} \partial_z s s^{-1}) - \frac{i}{24} \int_\Omega \mathrm{Tr}(d\tilde{s}\tilde{s}^{-1} \wedge [\tilde{s}\tilde{s}^{-1} \,\hat{,}\, d\tilde{s}\tilde{s}^{-1}]). \tag{44}
$$

Recall once more that s_P is the $S^\uparrow$-extension of s to the brane P_2 and $\tilde{s}$ is the extension of $s \cup s_P$ to the interior of the ball Ω. We shall show in Sect. 4, that the action (44) possesses a residual $SU(2)$ loop group symmetry.

3.4 Euclidean AdS_3 Branes

This case was not discussed in [11]. The reason is simple: the leading thread of our reasoning in this section is T-duality, while Ponsot, Schomerus and Teschner have organized their paper from the point of view of the residual symmetry of the D-brane configurations. They have considered only the case were this symmetry was semi-simple. However, the Euclidean AdS_3 branes, that we are going to study now, have a solvable residual symmetry. Consider thus the remaining case $M = AN$ embedded into $SL(2, \mathbf{C})$ in the standard way (cf. (16)):

$$b = \begin{pmatrix} e^{\phi/2} & 0 \\ \gamma e^{\phi/2} & e^{-\phi/2} \end{pmatrix}, \quad \phi \in \mathbf{R}, \ \gamma \in \mathbf{C}. \tag{45}$$

The possible first order D-branes are then the submanifolds Md of $SL(2, \mathbf{C})$ with d being a fixed element of $SL(2, \mathbf{C})$. As usual, we do not make here the most general choice of d; instead, we consider only the simplest possible d having the form

$$d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus our first order D-branes are three-dimensional submanifolds of $SL(2, \mathbf{C})$ of the form $p = b$ (cf. (45)), where $\phi \in \mathbf{R}$ and $\gamma \in \mathbf{C}$ vary.

3.4.1 The Target $D/G_1 = SL(2, \mathbf{C})/SU(2)$

Now we are ready to find the shape of the D-brane P_1 in D/G_1. It is given by the coset projection (26) of the first order D-brane (45):

$$h(p) \equiv h(b) = \begin{pmatrix} e^\phi & e^\phi \bar{\gamma} \\ e^\phi \gamma & e^\phi \gamma \bar{\gamma} + e^{-\phi} \end{pmatrix}, \quad \phi \in \mathbf{R}, \ \gamma \in \mathbf{C}. \tag{25}$$

Thus we immediately observe that the D-brane P_1 coincides with the whole target $SL(2, \mathbf{C})/SU(2)$. This is the reason why we call P_1 the Euclidean AdS_3 brane. In fact, whatever d we choose, it turns out that the corresponding D-brane P_1 sweeps the whole target space $SL(2, \mathbf{C})/SU(2)$. However, the choice of d has an influence on the shape of the dual D-brane P_2 in $SL(2, \mathbf{C})/SL(2, \mathbf{R})$.

As we have said, the first order action (10) together with the choice of the first order D-brane (45) (and $\alpha_D = 0$) determines the open string quintuple $(T_1, ds_1^2, H_1, P_1, \alpha_1)$. So far we have determined its four elements: the target $T_1 = SL(2, \mathbf{C})/SU(2)$ is parametrized as in (25), then

$$d\sigma_1^2 = -\frac{1}{4}(d\phi^2 + e^{2\phi}d\bar{\gamma}d\gamma), \qquad H_1 = -\frac{1}{2}e^{2\phi}d\phi \wedge d\bar{\gamma} \wedge d\gamma$$

and the D-brane P_1 coincides with the target T_1. It remains to determine α_1. We use the formula (13) and argue that the second term on its r.h.s. vanishes in this particular case. Indeed, we can choose the map $g_1(f_j)$ equal to $g_1(f_j) = 1$. Hence we conclude

$$\alpha_1 = \frac{1}{4}e^{2\phi}d\gamma \wedge d\bar{\gamma}.$$

The open string action therefore reads

$$S = -\frac{i}{2} \int_{S_\downarrow} d\bar{z} \wedge dz (\partial_z \phi \partial_{\bar{z}} \phi + e^{2\phi} \partial_z \gamma \partial_{\bar{z}} \bar{\gamma}). \tag{46}$$

The boundary conditions that accompany the action (46) say that the boundary of the world-sheet is not constrained but can be located everywhere on the AdS_3 target. We shall show in Sect. 4, that the action (46) possesses a residual AN current symmetry.

3.4.2 The Target $D/G_2 = SL(2, \mathbf{C})/SL^a(2, \mathbf{R})$

The derivation of the T-dual quintuple $(T_2, d\sigma_2^2, H_2, P_2, \alpha_2)$ is straightforward. We have to apply the coset projection map (34) to the first-order brane (45). We obtain

$$s(p) \equiv s(b) = \begin{pmatrix} 0 & 1 \\ 1 & \gamma + \bar{\gamma} \end{pmatrix}.$$

In the parametrization (32), the dual de Sitter brane P_2 is therefore just a line $w = 1, u = 0$. This means, in particular, that the two-form α_2 on it automatically vanishes.

Thus we conclude the discussion of the de Sitter dual of the Euclidean AdS_3 branes by writing for them the open string action:

$$S = -\frac{i}{4} \int_{S_\downarrow} d\bar{z} \wedge dz \, \mathrm{Tr}(\partial_{\bar{z}} ss^{-1} \partial_z ss^{-1}) - \frac{i}{24} \int_\Omega \mathrm{Tr}(d\tilde{s}\tilde{s}^{-1} \wedge [d\tilde{s}\tilde{s}^{-1} \hat{,} d\tilde{s}\tilde{s}^{-1}]).$$

Recall once more that s_P is the $S^\uparrow$-extension of s lying in the one-dimensional brane P_2 and $\tilde{s}$ is the extension of $s \cup s_P$ to the interior of the ball Ω. We shall show in Sect. 4, that this action possesses a residual AN loop group symmetry.

4 D-Branes and the Residual Loop Symmetry

In Sect. 3, we have described three types of branes in the $SL(2, \mathbf{C})/SU(2)$ WZW model and in its de Sitter dual. We have referred to them, respectively, as AdS_2, spherical and Euclidean AdS_3 branes. We shall now argue that each type has a residual loop group symmetry respecting the D-branes boundary conditions (P, α). We say *residual*, because there is even bigger loop group symmetry of the bulk model. We shall first describe it and then discuss the symmetry of various D-brane configurations.

4.1 Bulk Loop Group Symmetry

Consider antiholomorphic maps $g(\bar{z})$ from the Riemann sphere without poles into the complex group $SL(2, \mathbf{C})$. The set of such maps form a loop group $LSL(2, \mathbf{C})$. The $LSL(2, \mathbf{C})$ loop group symmetry of the first order bulk action

$$S = \frac{1}{2} \int_{\partial \Omega} d\sigma \wedge d\tau \, \mathrm{Im}\,\mathrm{Tr}(\partial_\tau ll^{-1} \partial_\sigma ll^{-1}) + \frac{1}{12} \int_\Omega \mathrm{Im}\,\mathrm{Tr}(dll^{-1} \wedge [dll^{-1} \hat{,} dll^{-1}])$$

$$- \frac{1}{2} \int_{\partial \Omega} d\sigma \wedge d\tau \, \mathrm{Im}\,\mathrm{Tr}(\partial_\sigma ll^{-1} i \partial_\sigma ll^{-1}) \tag{14}$$

is the direct consequence of the Polyakov–Wiegmann formula (11). Indeed, it is the matter of an easy check to see that the bulk action (14) does not change its value upon the replacing a configuration $l(z, \bar{z})$ by $g(\bar{z})l(z, \bar{z})$. In particular, also the field equations following from (14)

$$l^{-1} \partial_z l = 0$$

are manifestly $LSL(2, \mathbf{C})$-symmetric since a solution $l(\bar{z})$ becomes clearly another solution upon the transformation

$$l(\bar{z}) \rightarrow g(\bar{z})l(\bar{z}), \quad g(\bar{z}) \in LSL(2, \mathbf{C}). \tag{47}$$

The $LSL(2, \mathbf{C})$ symmetry transformation shows up also in the second order formalism. For concreteness, consider the target $SL(2, \mathbf{C})/SU(2)$ and the bulk σ-model (22)

$$S = -\frac{i}{4} \int_{\partial \Omega} d\bar{z} \wedge dz \, \mathrm{Tr}(\partial_{\bar{z}} hh^{-1} \partial_z hh^{-1}) - \frac{i}{24} \int_{\Omega} \mathrm{Tr}(d\tilde{h}\tilde{h}^{-1} \wedge [d\tilde{h}\tilde{h}^{-1} \, \hat{,} \, d\tilde{h}\tilde{h}^{-1}]). \tag{22}$$

Recall once more that $\tilde{h}$ is the extension of h to the interior of the ball Ω. First note that from the first order configuration l we obtain the second order trajectory f_1 by the (Iwasawa) decomposition (6). This means (cf. (19)) that

$$h = f_1 f_1^{\dagger} = ll^{\dagger} \tag{19b}$$

and the transformation (47) translates into

$$h(z, \bar{z}) \rightarrow g(\bar{z})h(z, \bar{z})g^{\dagger}(z). \tag{48}$$

Then note that the Polyakov–Wiegmann formula (11) holds note only for the indefinite metric $(., .)_{\mathcal{D}}$ but also for every invariant bilinear form on $Lie(D)$, in particular for Tr appearing in (22). Using this fact, it is the matter of an easy calculation to see that the action (22) is indeed invariant with respect to the loop group transformation (48).

In the de Sitter dual, the bulk action

$$S = -\frac{i}{4} \int_{S_{\downarrow}} d\bar{z} \wedge dz \, \mathrm{Tr}(\partial_{\bar{z}} ss^{-1} \partial_z ss^{-1}) - \frac{i}{24} \int_{\Omega} \mathrm{Tr}(d\tilde{s}\tilde{s}^{-1} \wedge [d\tilde{s}\tilde{s}^{-1} \, \hat{,} \, d\tilde{s}\tilde{s}^{-1}]) \tag{23}$$

has the symmetry

$$s(z, \bar{z}) \rightarrow g(\bar{z})s(z, \bar{z})g^{\dagger}(z)$$

which originates from the first order transformation (47) by the coset projection map $s = l\sigma_1 l^{\dagger}$ (cf. (34)).

4.2 Residual Loop Group Symmetry

4.2.1 First Order Formalism

Consider the first order brane $\Pi = Md$ in the double $D = SL(2, \mathbf{C})$, where M is some maximally isotropic subgroup of D. We know that the open string first-order action reads

(cf. (10))

$$S[l] = \frac{1}{2} \int_{S_\downarrow} d\sigma \wedge d\tau \operatorname{Im} \operatorname{Tr}(\partial_\tau l l^{-1} \partial_\sigma l l^{-1}) - \frac{1}{2} \int_{S_\downarrow} d\sigma \wedge d\tau \operatorname{Im} \operatorname{Tr}(\partial_\sigma l l^{-1} i \partial_\sigma l l^{-1})$$

$$+ \frac{1}{12} \int_\Omega \operatorname{Im} \operatorname{Tr}(d\tilde{l}\tilde{l}^{-1} \,\hat{\wedge}\, [d\tilde{l}\tilde{l}^{-1} \,\hat{\wedge}\, d\tilde{l}\tilde{l}^{-1}]), \tag{49}$$

where l is the true open string configuration defined on $S_\downarrow$ whose boundary $l(|z| = 1)$ takes values in $Md \subset D$, l_Π is its $S^\uparrow$ extension into the first order brane $\Pi = Md$ and $\tilde{l}$ is the extension of $l \cup l_\Pi$ to the ball Ω.

The open string action (49) cannot be symmetric with respect to all $LSL(2, \mathbf{C})$ loop transformations $l(z, \bar{z}) \to g(\bar{z})l(z, \bar{z})$ ($|z| \leq 1$). Indeed, an arbitrary $LSL(2, \mathbf{C})$ element $g(\bar{z})$ do not preserve the first-order brane boundary condition saying that $l(|z| = 1) \in Md$. On the other hand, we can consider the residual subgroup $LM_0 \subset LSL(2, \mathbf{C})$ consisting of the elements $g(\bar{z})$ satisfying $g(|z| = 1) \in M$ and having the property that the induced mapping from the equator of the Riemann sphere into the group M is homotopically trivial (of course, this is always true if the group M is simply connected). The action $l \to ml$ of this residual subgroup transforms one open string configuration $l(z, \bar{z})$ into another one $m(\bar{z})l(z, \bar{z})$ while respecting the boundary conditions $(ml)(|z| = 1) \in Md$. It makes therefore sense to ask whether the action (49) changes upon such a transformation.

In order to answer this question, we must extend the transformed configuration ml from $S_\downarrow$ to $S^\uparrow$ in such a way that the $S^\uparrow$ piece lies in the first order brane $\Pi = Md$. This is easy, we first consider the original extension l_Π of l and then any map $m_\Pi : S^\uparrow \to M$ satisfying $m_\Pi(|z| = 1) = m(|z| = 1)$, where $m(\bar{z}) \in LM_0$ (the map m_Π always exists due to our assumption about the homotopical triviality). Thus we set $(ml)_\Pi = m_\Pi l_\Pi$. In the analogous way, the tilde-extension of the transformed configuration ml can be written as $\widetilde{(ml)} = \tilde{m}\tilde{l}$ for an appropriate $\tilde{m}$. Now we replace l, l_Π and $\tilde{l}$ in (49) by ml, $m_\Pi l_\Pi$ and $\tilde{m}\tilde{l}$ and use the Polyakov–Wiegman formula (11). The result is immediate:

$$S[ml] = S[l] - \frac{1}{2} \int_{S^\uparrow} \operatorname{Im} \operatorname{Tr}(m_\Pi^{-1} dm_\Pi \wedge dl_\Pi l_\Pi^{-1}). \tag{50}$$

Let us argue that the second term in the r.h.s. of (50) vanishes therefore the action (49) is invariant with respect to the residual loop group symmetry LM_0. Indeed, this follows from the isotropy of the $Lie(M)$ with respect to the $\operatorname{Im} \operatorname{Tr}$ and from the fact that both forms $m_\Pi^{-1} dm_\Pi$ and $dl_\Pi l_\Pi^{-1}$ are $Lie(M)$-valued.

4.2.2 Second Order Formalism

In principle, the second order formalism follows from the first order one and, therefore, we can consider that we have already proven in the precedent Sect. 4.2.1 the residual loop group symmetry of all D-brane configurations considered in this paper. On the other hand, the reader might wish to see the proof of the residual symmetry by working directly in the second order formalism. In three cases out of six, we do not have a coordinate free expression for the two-form α on the brane. Due to this circumstance, we have to work in coordinates, the relevant formulae for the loop group action are quite complicated and we do not detail the second-order symmetry demonstration here. However, in the three remaining cases (including all brane configurations considered by Ponsot, Schomerus and Teschner [11]) we do have the coordinate free expression for the form α and this fact makes possible to

give the simple second-order proof of the residual loop symmetry. As an example, we prove the residual $SU(2)$ loop symmetry of the spherical brane in the $SL(2, \mathbf{C})/SU(2)$ background. Recall that its action principle reads

$$S = -\frac{i}{4} \int_{S_\downarrow} d\bar{z} \wedge dz \, \mathrm{Tr}(\partial_{\bar{z}} hh^{-1}\partial_z hh^{-1})$$

$$-\frac{i}{24} \int_\Omega \mathrm{Tr}(d\tilde{h}\tilde{h}^{-1} \wedge [d\tilde{h}\tilde{h}^{-1} \overset{\wedge}{,} d\tilde{h}\tilde{h}^{-1}]) + i\frac{\cosh 2\rho}{8 \sinh^2 2\rho} \int_{S^\uparrow} \mathrm{Tr}(h_P dh_P \wedge dh_P). \quad (43)$$

Recall once more that h_P is the $S^\uparrow$-extension of h lying in the brane P_1 and $\tilde{h}$ is the extension of $h \cup h_P$ to the interior of the ball Ω. The boundary conditions that accompany the action (43) say that the boundary of the world-sheet lies in the D-brane (41).

We know from the first order analysis that the residual symmetry group is $LSU(2)$, i.e. the group of maps $g(\bar{z})$ such that $g(|z| = 1) \in SU(2)$. We shall denote the elements of $LSU(2)$ as $m(\bar{z})$ and, as in the precedent subsection, to every $m(\bar{z})$ we associate also the configurations m_Π and $\tilde{m}$. We remark that $m_\Pi^\dagger = m_\Pi^{-1}$ and we use the Polyakov–Wiegmann formula (11) to calculate the transformed action

$$S[mhm^\dagger] = S[h] + \frac{i}{4} \int_{S^\uparrow} \mathrm{Tr}\{m_\Pi^{-1}dm_\Pi \wedge (h_P^{-1}dh_P + dh_P h_P^{-1})$$

$$- m_\Pi^{-1}dm_\Pi \wedge h_P m_\Pi^{-1}dm_\Pi h_P^{-1}\}$$

$$+ i\frac{\cosh 2\rho}{8 \sinh^2 2\rho} \int_{S^\uparrow} \mathrm{Tr}(m_\Pi h_p m_\Pi^{-1} d(m_\Pi h_p m_\Pi^{-1}) \wedge (m_\Pi h_p m_\Pi^{-1})).$$

At a first sight, it is not evident that the contributions from the second and third line cancel each other. However, it is indeed true due to the identity

$$h_P^{-1} = 2\cosh 2\rho - h_P \quad (51)$$

holding on the surface of the D-brane $P_1 = \{h; \mathrm{Tr}(h) = 2\cosh 2\rho\}$.

The identity (51) has its cousins for all other branes for which the form α can be written in the coordinate free way. We list them in order to facilitate the work of the reader who wishes to check the second order residual symmetry also for them. It is

$$(h\sigma_1)^{-1} = h\sigma_1 - c, \qquad \mathrm{Tr}(h\sigma_1) = c,$$

for the AdS_2 branes in the $SL(2, \mathbf{C})/SU(2)$ WZW model and

$$s^{-1} = s, \qquad \mathrm{Tr}(s) = 0$$

for the spherical branes in the de Sitter background. We recall that s and h are Hermitean matrices, $\det h = 1$, $\det s = -1$ and $\mathrm{Tr}\, h$ is positive.

5 Conclusions and Outlook

The possibility to formulate the $SL(2, \mathbf{C})/SU(2)$ WZW model in the first order way (14) was a fruit of the Poisson–Lie symmetry of this theory [9]. This property then leads to the T-dualizability of the model with the T-dual being the de Sitter space. In this paper, we have

applied the general theory [7, 8] of the D-branes Poisson–Lie T-duality to this particular example. We stress, however, that our results may find applications also in the direct study of the $SL(2, \mathbf{C})/SU(2)$ WZW model without concentrating on the T-duality story. In particular, we have been able to formulate the dynamics of the AdS_2 brane in the coordinate independent manner (31) which could render more transparent the geometrical structures involved in this system. As it is well-known [13–22] several technical issues concerning string propagation in the Lorentzian AdS_3 can be Wick-continued to the Euclidean case treated here. It is an interesting open question how the T-duality discovered in the Euclidean context manifests itself in the Lorentzian one.

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